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Research Article

Bayesian Analysis of Two Parameter Weibull Distribution Using Different Loss Functions

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Abstract:

This paper focuses on the Bayesian technique to estimate the parameters of the Weibull distribution. At this location, we use both informative and non-informative priors. We calculate the estimators and their posterior risks using different asymmetric and symmetric loss functions. Bayes estimators do not have a closed form under these loss functions. Therefore, we use an approximation approach established by Lindley to get the Bayes estimates. A comparative analysis is conducted to compare the suggested estimators using Monte Carlo simulation based on the related posterior risk. We also analyze the impact of distinct loss functions when using various priors.

Keywords: Bayesian Estimation; Maximum Likelihood Estimation Lindley Approximation; Monte Carlo Simulation; Weibull Distribution.
Dataset link: -

1. Introduction

The Weibull distribution, established by [1], is the most widely used life-time distribution. This distribution is often utilized in dependability and risk assessment because to its flexibility to change form based on factors. This distribution has been extensively used in sectors such as renewable energy, geothermal energy, medicinal, biological, environmental, and earth sciences [2].

The probability density function for the two parameter Weibull distribution with scale and shape parameters α and β respectively both positive is given by.

$$f(x;\alpha,\beta)=\frac{\alpha}{\beta^{\alpha}}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\quad 0<x<\infty\quad \alpha>0,\beta>0 \tag{1}$$

The two parameter Weibull distribution may have an increasing or decreasing hazard function based on the shape parameter α value. The skewness of this distribution makes it a good starting point for modelling purposes monotone hazard rates. This distribution is excellent for modelling extreme value data.

Several researches have been conducted to determine the optimal estimate method for Weibull distribution. Although there is much literature on estimating Weibull distributions using the frequentist technique, researchers have. Recently, substantial emphasis has been paid to Bayesian inference of Weibull parameters. [3] used the Gamma prior for censored data from the two-parameter Weibull distribution. Using Jeffrey and the extended Jeffrey before. [4] compared Bayesian estimate to MLE for the Weibull distribution. [5] compared the Weibull distribution's scale parameter to the maximum likelihood estimator when using the LINEX loss function. [6] compared the Bayesian method for Weibull model parameters with progressive censoring, which uses a bivariate prior distribution under squared error loss, to the maximum likelihood method. [7] developed Bayes estimators for the Weibull distribution with two parameters and three loss functions, extending Jeffrey's priors. [8] conducted a Bayesian investigation on the two-parameter Weibull distribution with different loss functions and priors. [9] simulated the Bayes estimator for parameters in the Weibull distribution using Gamma priors and the squared error loss function. They used Lindley's approximation method to get approximate Bayesian estimators for parameters that cannot be expressed explicitly. [10] estimated the scale parameter of the Weibull distribution using inverse gamma and Jeffrey's prior. [11] performed a comparison of Bayesian and classical parameter estimate methods for the Weibull distribution. He used nine classical methods to estimate parameters and two Bayesian methods (Lindley approximation and Tierney Kadane approximation) to generate Bayesian estimators for various symmetric and asymmetric loss functions. He recommended using the Lindley approximation method to compute Bayesian estimators.

A Bayes decision becomes a Bayes estimator when it minimizes the risk function [12]–[14]. The best option is one that has the least amount of future risk. The decision-theoretical approach considers extra information about the potential outcomes of choices. Choosing a good loss function is critical in decision theory. To derive Bayes estimators, we investigate several loss functions. Again, under the Bayesian method, the main challenge for a particular model depending on the previous distribution and loss functions used. We employ various priors, both informative and non-informative, to calculate estimators and posterior risks for various loss functions.

The remaining portions of the paper are organized as follows. Section 2 discusses the maximum likelihood method of calculating Weibull parameters. Section 3 provides a posterior distribution based on informative and non-informative priors. In section 4, we use Lindley's approximation approach to produce Bayes estimators for symmetric and asymmetric loss functions with varying priors. Section 5 includes simulated experiments comparing the performance of various estimators based on posterior risk. This study examines the impact of various loss functions using informative and non-informative priors. Section 6 presents some conclusions.

2. Method:

Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ denotes a random sample of size n drawn from Weibull distribution with probability density Equation (1)

Then the likelihood function is.

$$L(\alpha,\beta) = \frac{\delta^n}{\alpha^{n\delta}}\left(\prod_{i=1}^{n} x_i\right)^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} \tag{2}$$

so that the log-likelihood equation can be defined as

$$lnL = nln\delta - n\delta ln\alpha + (\delta-1)\sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} \tag{3}$$

Under differentiation of Equation (2) with respect to α and δ and equating to zero, Equation (3) becomes

$$\frac{\partial lnL}{\delta} = \frac{n}{\delta} - nln\alpha + \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} ln\frac{x_i}{\alpha} = 0 \tag{4}$$

$$\frac{\partial lnL}{\partial\alpha} = \frac{-n\delta}{\alpha} + \frac{\delta}{\alpha}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} = 0 \tag{5}$$

This system of equations is infeasible to obtain their explicit solutions. In this situation, the ML estimates of α and δ are computed by using "MASS" package in RStudio. Nelder-Mead optimization method is employed for solving the equations analytically

Priors and Posterior Distributions

Recently, Bayesian approach became more popular for analysing failure time data. If the prior knowledge of the parameter is in hand it is better to make use of an informative prior. Otherwise, make use of a prior which is non-informative. Posterior distribution is the combination of sample information within the likelihood function and the probabilistic information about the parameters available as prior distribution. Here, we consider Weibull model as the sampling distribution, for deriving posterior distribution it is mingled with both informative and non-informative priors.

Posterior Distribution using Non-Informative Prior

Here, we use [15] prior based on Fisher's information given by

$$\pi(\gamma) \propto \sqrt{I(\gamma)}.$$

[16] deals with the non-informative prior for the two parameter Weibull distribution given by

$$\pi_1(\alpha, \delta) \propto \left(\frac{1}{\alpha\delta}\right).$$

Then the joint posterior distribution is given by

$$\pi_1(\alpha, \delta \mid x) = \frac{1}{K_1}\left(\frac{1}{\alpha\delta}\right)\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}}; \alpha, \delta > 0 \tag{6}$$

where

$$K_1 = \int_0^\infty \int_0^\infty \left(\frac{1}{\alpha\delta}\right)\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} d\alpha d\delta \tag{7}$$

Posterior Distribution using Informative Priors

Here, we consider two informative priors viz. Gamma Prior and Gumbel type-II Prior for both α and δ.

Posterior Distribution using Gamma Prior

Here, we consider independent gamma priors for both α and δ of Weibull distribution. That is $\alpha \sim$ gamma (a_2, b_2) and $\delta \sim$ gamma (a_1, b_1).

The joint prior distribution of α and δ is given by

$$\pi_2(\alpha, \delta) \propto \delta^{a_1-1} e^{-b_1\delta} \alpha^{a_2-1} e^{-b_2\alpha}; \alpha, \delta, a_1, b_1, a_2, b_2 > 0 \text{ and thereby} \tag{8}$$

the joint posterior distribution of α and δ is

$$\pi_2(\alpha, \delta \mid x) = \frac{1}{K_2}\delta^{a_1-1} e^{-b_1\delta} \alpha^{a_2-1} e^{-b_2\alpha}\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} \tag{9}$$

where

$$K_2 = \int_0^\infty \int_0^\infty \delta^{a_1-1} e^{-b_1\delta} \alpha^{a_2-1} e^{-b_2\alpha}\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} d\alpha d\delta \tag{10}$$

Posterior Distribution using Gumbel type-II Prior

We propose here that δ and α follow independent Gumbel type II priors having probability density functions

$$\pi_3(\alpha, \delta) \propto \frac{e^{\left(\frac{-a_3}{\delta}\right)}}{\delta^2}\frac{e^{\left(\frac{-b_3}{\alpha}\right)}}{\alpha^2}; \alpha, \delta, a_3, b_3 > 0$$

$$\pi(\alpha) = \frac{b_3}{\alpha^2} e^{-\left(\frac{b_3}{\alpha}\right)}; \alpha, b_3 > 0$$

respectively. Then the joint prior distribution is given by

$$\pi_3(\alpha,\delta) \propto \frac{e^{\left(\frac{-a_3}{\delta}\right)}}{\delta^2} \frac{e^{\left(\frac{-b_3}{\alpha}\right)}}{\alpha^2}; \alpha, \delta, a_3, b_3 > 0 \quad (11)$$

the joint posterior distribution of the parameters α and δ is

$$\pi_3(\alpha,\delta \mid x) = \frac{1}{K_3} \frac{e^{\left(\frac{-a_3}{\delta}\right)}}{\delta^2} \frac{e^{\left(\frac{-b_3}{\alpha}\right)}}{\alpha^2} \left(\frac{\delta^n}{\alpha^{n\delta}}\right) \prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} \quad (12)$$

where

$$\mathrm{K}_3 = \int_0^\infty \int_0^\infty \frac{e^{\left(\frac{-a_3}{\delta}\right)}}{\delta^2} \frac{e^{\left(\frac{-b_3}{\alpha}\right)}}{\alpha^2} \left(\frac{\delta^n}{\alpha^{n\delta}}\right) \prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} d\alpha\, d\delta \quad (13)$$

3. Results and Discussion

Bayes Estimators and Posterior Risks under Different Loss Functions

In this section, we used asymmetric and symmetric loss functions to generate Bayes estimators and posterior risks for the unknown parameters of the Weibull distribution. Bayes estimators and their associated hazards are calculated using three distinct loss functions: SELF (squared error loss function), PLF (precautionary loss function), and LINEX (linear exponential loss function). For further information, go to [17]–[19].

Bayes estimators and their posterior risks are explained in **Table 1** for the unknown parameter α under mentioned loss functions.

Table 1. Bayes estimators and posterior risks under different loss functions

Loss Function	Bayes Estimator (BE)	Posterior Risk (PR)
$SELF = (d-\gamma)^2$	$E(\gamma \mid x)$	$E(\gamma^2 \mid x) - [E(\gamma \mid x)]^2$
$PLF = \frac{(d-\gamma)^2}{d}$	$\sqrt{E(\gamma^2 \mid x)}$	$2\left[\sqrt{E(\gamma^2 \mid x)} - E(\gamma \mid x)\right]$
$LINEX = e^{m(d-\gamma)} - m(d-\gamma) - 1$	$\frac{-1}{m}\log\left[E(e^{-m\gamma \mid x})\right]$	$\log\left[E(e^{-m\gamma \mid x})\right] + mE(\gamma \mid x)$

Lindley Approximation

The joint posterior distribution of α and δ in Equations (12) and (13) is a ratio of two integrals that cannot be solved analytically.

The posterior expectation may be stated as:

$$I(x) = E[u(\alpha,\delta) \mid x] = \frac{\int u(\alpha,\delta)\exp\left[L(\alpha,\delta) + \rho(\alpha,\delta)\right]d(\alpha,\delta)}{\int \exp\left[L(\alpha,\delta) + \rho(\alpha,\delta)\right]d(\alpha,\delta)} \quad (14)$$

where $u(\alpha,\delta)$ is a function of α and δ only, $L(\alpha,\delta)$ is the log-likelihood and $\rho(\alpha,\delta)$ is the log of joint prior of α and δ.

According to [20], if the sample size n is sufficiently large, the above equation can be approximately evaluated as:

$$I(x) = u(\hat{\alpha},\hat{\delta}) + 0.5[(\hat{u}_{\alpha\alpha} + 2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha} + (\hat{u}_{\delta\alpha} + 2\hat{u}_{\delta}\hat{\rho}_{\alpha})\hat{\sigma}_{\delta\alpha} + (\hat{u}_{\alpha\delta} + 2\hat{u}_{\alpha}\hat{\rho}_{\delta})\hat{\sigma}_{\alpha\delta}$$
$$+(\hat{u}_{\delta\delta} + 2\hat{u}_{\delta}\hat{\rho}_{\delta})\hat{\sigma}_{\delta\delta}] + 0.5\big[(\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{u}_{\delta}\hat{\sigma}_{\alpha\delta})\big(\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\alpha\delta\alpha}\hat{\sigma}_{\alpha\delta} + \hat{L}_{\delta\alpha\alpha}\hat{\sigma}_{\delta\alpha}$$
$$+\hat{L}_{\delta\delta\alpha}\hat{\sigma}_{\delta\delta}\big) + (\hat{u}_{\alpha}\hat{\sigma}_{\delta\alpha} + \hat{u}_{\delta}\hat{\sigma}_{\delta\delta})\big(\hat{L}_{\delta\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\alpha\delta\alpha}\hat{\sigma}_{\alpha\delta} + \hat{L}_{\delta\alpha\delta}\hat{\sigma}_{\delta\alpha} + \hat{L}_{\delta\delta\delta}\hat{\sigma}_{\delta\delta}\big)\big]$$

where $\hat{\alpha}$ and $\hat{\delta}$ are the MLE's of α and δ respectively.

$$\hat{u}_{\alpha} = \frac{\partial u(\hat{\alpha},\hat{\delta})}{\partial \alpha}; \hat{u}_{\delta} = \frac{\partial u(\hat{\alpha},\hat{\delta})}{\partial \delta}; \hat{u}_{\alpha\alpha} = \frac{\partial^2 u(\hat{\alpha},\hat{\delta})}{\partial \alpha^2}; \hat{u}_{\delta\delta} = \frac{\partial^2 u(\hat{\alpha},\hat{\delta})}{\partial \delta^2}; \hat{u}_{\alpha\delta} = \frac{\partial^2 u(\hat{\alpha},\hat{\delta})}{\partial \alpha\, \partial \delta}$$

$$\hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha}\hat{\delta})}{\partial \alpha^3}; \hat{L}_{\delta\delta\delta} = \frac{\partial^3 L(\hat{\alpha},\hat{\delta})}{\partial \delta^3}; \hat{L}_{\alpha\delta} = \frac{\partial^2 L(\hat{\alpha},\hat{\delta})}{\partial \alpha\, \partial \delta}, \text{ and so on}$$

Since, α and δ are independent $\hat{\sigma}_{\alpha\delta} = 0$.

With the above defined expressions, the values of the estimates for Weibull distribution are as follows.

$$E[u(\alpha,\delta) \mid x] = u(\hat{\alpha},\hat{\delta}) + 0.5[(\hat{u}_{\alpha\alpha}\hat{\sigma}_{\alpha\alpha}) + (\hat{u}_{\delta\delta}\hat{\sigma}_{\delta\delta})] + \hat{u}_{\alpha}\hat{\rho}_{\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{u}_{\delta}\hat{\rho}_{\delta}\hat{\sigma}_{\delta\delta}$$
$$+ 0.5[(\hat{u}_{\alpha}\hat{\sigma}^2_{\alpha\alpha}L_{\alpha\alpha\alpha}) + (\hat{u}_{\delta}\hat{\sigma}^2_{\delta\delta}L_{\delta\delta\delta})] \quad (15)$$

where $L(\alpha,\delta)$ is the log-likelihood equation (2). Also

$$L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2} = n\left(\frac{\delta}{\alpha^2}\right) - \left(\frac{\delta}{\alpha}\right)^2 \sum_{i=1}^{n} \left(\frac{x_i}{\alpha}\right)^{\delta} - \frac{\delta}{\alpha^2}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}, \hat{\sigma}_{\alpha\alpha} = -\frac{1}{L_{\alpha\alpha}} \quad (16)$$

$$L_{\alpha\alpha\alpha} = \frac{\partial^3 L}{\partial \alpha^3} = -2n\left(\frac{\delta}{\alpha^3}\right) + \left(\frac{\delta^3}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} + 2\left(\frac{\delta^2}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}$$
$$+\left(\frac{\delta^2}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} + 2\left(\frac{\delta}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}, \quad (17)$$

$$L_{\delta\delta} = \frac{\partial^2 L}{\partial \delta^2} = \frac{-n}{\delta^2} - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}\left(\log\left(\frac{x_i}{\alpha}\right)\right)^2, \quad (18)$$

$$\hat{\sigma}_{\delta\delta} = -\frac{1}{L_{\delta\delta}}$$

$$L_{\delta\delta\delta} = \frac{\partial^3 L}{\partial \delta^3} = \frac{2n}{\delta^3} - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}\left(\log\left(\frac{x_i}{\alpha}\right)\right)^3. \quad (19)$$

The log joint prior density of Jeffrey's prior is

$$\rho(\alpha,\delta) = -\log(\alpha) - \log(\delta),$$
$$\rho_{\alpha} = \frac{\partial \rho(\alpha,\delta)}{\partial \alpha} = \frac{-1}{\alpha} \text{ and}$$
$$\rho_{\delta} = \frac{\partial \rho(\alpha,\delta)}{\partial \delta} = \frac{-1}{\delta}.$$

The log joint prior density of Gamma prior is as follows:

$$\rho(\alpha,\delta) = (a_1 - 1)\log(\delta) - b_1\delta + (a_2 - 1)\log(\alpha) - b_2\alpha,$$
$$\rho_{\alpha} = \frac{\partial \rho(\alpha,\delta)}{\partial \alpha} = \frac{a_2 - 1}{\alpha} - b_2 \text{ and}$$
$$\rho_{\delta} = \frac{\partial \rho(\alpha,\delta)}{\partial \delta} = \frac{a_1 - 1}{\delta} - b_1.$$

Again, the log of the joint prior density of Gumbel type-II prior is

$$\rho(\alpha,\delta) = \frac{-a_3}{\delta} - \log(\delta^2) - \frac{-b_3}{\alpha} - \log(\alpha^2),$$
$$\rho_\alpha = \frac{\partial\rho(\alpha,\delta)}{\partial\alpha} = \frac{b_3 - 2\alpha}{\alpha^2} \text{ and}$$
$$\rho_\delta = \frac{\partial\rho(\alpha,\delta)}{\partial\delta} = \frac{a_3 - 2\delta}{\delta^2}.$$

Table 2. shows the $u(\alpha)$ and $u(\delta)$ functions under the mentioned loss functions

Loss Function	$u(\alpha)$	u_α	$u_{\alpha\alpha}$	$u(\delta)$	u_δ	$u_{\delta\delta}$
SELF	α	1	0	δ	1	0
PLF	α^2	2α	2	δ^2	2δ	2
LINEX	$e^{-m\alpha}$	$-me^{-m\alpha}$	$m^2e^{-m\alpha}$	$e^{-m\delta}$	$-me^{-m\delta}$	$m^2e^{-m\delta}$

We derive the estimates of α and δ under the mentioned loss functions using Jeffrey's, Gamma and Gumbel type-II prior as follows.

Lindley's Approximation of α and δ using SELF

The approximate Bayes estimates of δ using Jeffrey's, Gamma and Gumbel type-II priors under SELF are obtained respectively as follows.

$$\hat{\delta}_{JS} = \hat{\delta} + \left(\frac{-1}{\hat{\delta}}\right)\sigma_{\delta\delta} + 0.5(L_{\delta\delta\delta}\sigma^2_{\delta\delta})$$

$$\hat{\delta}_{GS} = \hat{\delta} + \left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} + 0.5(L_{\delta\delta\delta}\sigma^2_{\delta\delta}) \text{ and}$$

$$\hat{\delta}_{GUS} = \hat{\delta} + \left(\frac{a_3 - 2\hat{\delta}}{\hat{\delta}^2}\right)\sigma_{\delta\delta} + 0.5(L_{\delta\delta\delta}\sigma^2_{\delta\delta}) \text{ respectively}$$

The posterior risks of δ using Jeffrey's, Gamma and Gumbel type-II priors under SELF are obtained and they are

$$R(\hat{\delta}_{JS}) = \hat{\delta}^2 - \sigma_{\delta\delta} + (\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta}) - (\hat{\delta}_{JS})^2,$$

$$R(\hat{\delta}_{GS}) = \hat{\delta}^2 + 0.5(2\sigma_{\delta\delta}) + 2\hat{\delta}\left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta}) - (\hat{\delta}_{GS})^2 \text{ and}$$

$$(4.17)\ R(\hat{\delta}_{GUS}) = \hat{\delta}^2 + 0.5(2\sigma_{\delta\delta}) + 2\hat{\delta}\left(\frac{a_3 - 2\hat{\delta}}{\hat{\delta}^2}\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta}) - (\hat{\delta}_{GUS})^2$$

The approximate Bayes estimates of α using Jeffrey's, Gamma and Gumbel type-II priors under SELF are in the following form.

$$\hat{\alpha}_{JS} = \hat{\alpha} + \left(\frac{-1}{\hat{\alpha}}\right)\sigma_{\alpha\alpha} + 0.5(L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}),$$

$$\hat{\alpha}_{GS} = \hat{\alpha} + \left(\frac{a_2 - 1}{\hat{\alpha}} - b_2\right)\sigma_{\alpha\alpha} + 0.5(L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}) \text{ and}$$

$$\hat{\alpha}_{GUS} = \hat{\alpha} + \left(\frac{b_3 - 2\hat{\alpha}}{\hat{\alpha}^2}\right)\sigma_{\alpha\alpha} + 0.5(L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}).$$

Also, the posterior risks of α using Jeffrey's, Gamma and Gumbel type-II priors under SELF are obtained respectively as follows.

$$R(\hat{\alpha}_{JS}) = \hat{\alpha}^2 + \sigma_{\alpha\alpha} - 2\sigma_{\alpha\alpha} + \hat{\alpha}L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha} - (\hat{\alpha}_{JS})^2,$$

$$R(\hat{\alpha}_{GS}) = \hat{\alpha}^2 + 0.5(2\sigma_{\alpha\alpha}) + 2\hat{\alpha}\left(\frac{a_2 - 1}{\hat{\alpha}} - b_2\right)\sigma_{\alpha\alpha} + 0.5(2\hat{\alpha}L_{\alpha\alpha\alpha}\sigma_{\alpha}\alpha^2) - (\hat{\alpha}_{GS})^2 \text{ and}$$

$$R(\hat{\alpha}_{GUS}) = \hat{\alpha}^2 + 0.5(2\sigma_{\alpha\alpha}) + 2\hat{\alpha}\left(\frac{b_3 - 2\hat{\alpha}}{\hat{\alpha}^2}\right)\sigma_{\alpha\alpha} + 0.5(2\hat{\alpha}L_{\alpha\alpha\alpha}\sigma_{\alpha\alpha}^2) - (\hat{\alpha}_{GUS})^2.$$

Lindley's Approximation of α and δ using PLF

The approximate Bayes estimates of δ using Jeffrey's, Gamma and Gumbel type-II priors under PLF are obtained ad they are

$$(4.24)\ \hat{\delta}_{JP} = \left(\hat{\delta}^2 + \sigma_{\delta\delta} + \hat{\delta}L_{\delta\delta\delta}\sigma_{\delta\delta}^2\right)^{\frac{1}{2}} \qquad (13)$$

$$(4.25)\ \hat{\delta}_{GP} = \left(\hat{\delta}^2 + 0.5(2\sigma_{\delta\delta}) + 2\hat{\delta}^2\left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma_{\delta\delta}^2\right)^{\frac{1}{2}}$$

$$(4.26)\ \delta_{GUP} = \left(\hat{\delta}^2 + 0.5(2\sigma_{\delta\delta}) + 2\hat{\delta}\left(\frac{a_3 - 2\hat{\delta}^2}{\hat{\delta}}\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma_{\delta\delta}^2\right)^{\frac{1}{2}} \text{ respectively}$$

By using the similar procedure, the approximate Bayes estimates and posterior risks of α using Jeffrey's, Gamma and Gumbel type-II priors under PLF are obtained.

Lindley's Approximation of α and δ using LINEX

The approximate Bayes estimates of δ using Jeffrey's, Gamma and Gumbel type-II priors under LINEX are obtained respectively as follows.

(4.27)

$$\hat{\delta}_{JL} = \frac{-1}{m}log[e^{-m\hat{\delta}} + 0.5\left(m^2e^{-m\hat{\delta}}\sigma_{\delta\delta}\right) + \left(\frac{1}{\hat{\delta}}\right)ae^{-m\hat{\delta}}\sigma_{\delta\delta} - 0.5\left(me^{-m\hat{\delta}}L_{\delta\delta\delta}\sigma_{\delta\delta}^2\right)],$$

(4.28)

$$\hat{\delta}_{GL} = \frac{-1}{m}log\left[e^{-m\hat{\delta}} + 0.5\left(m^2e^{-m\hat{\delta}}\sigma_{\delta\delta}\right) - me^{-m\hat{\delta}}\left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} - 0.5\left(me^{-m\hat{\delta}}L_{\delta\delta\delta}\sigma_{\delta\delta}^2\right)\right] \text{ and}$$

(4.29)

$$\hat{\delta}_{GUL} = \frac{-1}{m}log\left[e^{-m\hat{\delta}} + 0.5\left(m^2e^{-m\hat{\delta}}\sigma_{\delta\delta}\right) - me^{-m\hat{\delta}}\left(\frac{a_3 - 2\hat{\delta}}{\hat{\delta}^2}\right)\sigma_{\delta\delta} - 0.5\left(me^{-m\hat{\delta}}L_{\delta\delta\delta}\sigma_{\delta\delta}^2\right)\right].$$

Simulation Study

Here, we carried out a study to assess the performance of Bayesian estimators obtained in Section 4 through Monte Carlo simulation with respect to posterior risks for different sample sizes and values of parameters. All programs were written using the R software. Here, we choose samples of sizes, n=10, 50 and 100. The parameter values are α=0.5, 1.5 and δ=0.8, 1.2. Also, the values for the loss parameter are m = ±0.6 and ±1.6. This was iterated 5000 times.

Table 3. That the maximum likelihood estimates of δ is same for both $\alpha = 0.5$ and 1.5

n	$\bar{\alpha}$	δ	$\hat{\alpha}_{ML}$	$\hat{\delta}_{ML}$
10	0.5	0.8 1.2	0.3554 0.3982	0.7632 1.1449
	1.5	0.8 1.2	1.0662 1.1948	0.7632 1.1449
50	0.5	0.8 1.2	0.4435 0.4616	0.7959 1.1939
	1.5	0.8 1.2	1.3307 1.3847	0.7959 1.1939
100	0.5	0.8 1.2	0.4995 0.4997	0.816 1.2241

n	$\bar{\alpha}$	δ	$\hat{\alpha}_{ML}$	$\hat{\delta}_{ML}$
	1.5	0.8 1.2	1.4981 1.499	0.816 1.2241

It is clear from the Table 3 that the maximum likelihood estimates of δ is same for both α = 0.5 and 1.5. The Bayes estimators and corresponding posterior risks of α and δ are reported in Tables 4 to 9.

Table 4. Bayes estimators for δ(n=10) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.73832(0.02965)	0.71384(0.02783)	0.93253(0.00159)
		PLF	0.75814(0.03963)	0.73308(0.03847)	0.93338(0.00171)
		LINEX (m = 0.6)	0.72958(0.00524)	0.70576(0.00484)	0.93168(0.00050)
		LINEX (m = 1.6)	0.71594(0.03581)	0.69364(0.03231)	0.92832(0.00672)
		LINEX (m = −0.6)	0.74733(0.0054)	0.72243(0.00515)	0.93269(9.78e − 05)
		LINEX (m = −1.6)	0.76254(0.03874)	0.73754(0.03791)	0.93181(−0.00114)
	1.2	SELF	1.10758(0.06671)	1.00251(0.04783)	1.2819(0.04933)
		PLF	1.13730(0.05943)	1.02609(0.04716)	1.30101(0.03820)
		LINEX (m = 0.6)	1.08813(0.01167)	0.98955(0.00777)	1.26576(0.00969)
		LINEX (m = 1.6)	1.05920(0.07734)	0.97375(0.04601)	1.23341(0.07760)
		LINEX (m = −0.6)	1.12793(0.01220)	1.01825(0.00944)	1.29535(0.00805)
		LINEX (m = −1.6)	1.16184(0.08680)	1.05017(0.07625)	1.31195(0.04805)

Table 5. Bayes estimators for δ(n = 50) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.7912(0.00731)	0.78468(0.00721)	0.83420(0.00587)
		PLF	0.79589(0.00922)	0.78926(0.00916)	0.83771(0.00702)
		LINEX (m = 0.6)	0.78909(0.00131)	0.78253(0.00128)	0.82239(0.00108)
		LINEX (m = 1.6)	0.78550(0.00925)	0.77904(0.00902)	0.82920(0.00799)
		LINEX (m = −0.6)	0.79348(0.00132)	0.78686(0.00130)	0.83591(0.00103)
		LINEX (m = −1.6)	0.79715(0.00939)	0.79053(0.00935)	0.83858(0.00700)
	1.2	SELF	1.18697(0.01646)	1.16050(0.01539)	1.22528(0.01552)
		PLF	1.19389(0.01383)	1.16712(0.01323)	1.23160(0.01264)
		LINEX (m = 0.6)	1.18206(0.00294)	1.15598(0.00271)	1.22053(0.00284)
		LINEX (m = 1.6)	1.17408(0.02063)	1.14895(0.01848)	1.21234(0.02070)
		LINEX (m = −0.6)	1.19193(0.00297)	1.16521(0.00282)	1.22984(0.00273)
		LINEX (m = −1.6)	1.20015(0.02108)	1.17338(0.02060)	1.23697(0.01870)

Table 6. Bayes estimators for δ(n = 100) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.81315(0.00351)	0.80980(0.00348)	0.83263(0.00324)
		PLF	0.81531(0.00431)	0.81195(0.00429)	0.83458(0.00389)
		LINEX (m = 0.6)	0.81210(0.00063)	0.80870(0.00062)	0.83165(0.00058)
		LINEX (m = 1.6)	0.81036(0.00446)	0.80704(0.00440)	0.82997(0.00426)
		LINEX (m = −0.6)	0.81420(0.00063)	0.81085(0.00062)	0.83360(0.00057)
		LINEX (m = −1.6)	0.81597(0.00450)	0.81261(0.00449)	0.83515(0.00403)
	1.2	SELF	1.21982(0.0	1.20684(0.00762)	1.23715(0.00775)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
		PLF	1.22306(0.00674)	1.21000(0.00631)	1.24028(0.00626)
		LINEX (m = 0.6)	1.21746(0.00141)	1.20458(0.00135)	1.23481(0.00140)
		LINEX (m = 1.6)	1.21357(0.01000)	1.20094(0.00944)	1.23085(0.01008)
		LINEX (m = −0.6)	1.22220(0.00142)	1.20916(0.00138)	1.23946(0.00137)
		LINEX (m = −1.6)	1.22616(0.01013)	1.21309(0.00999)	1.24320(0.00967)

Table 7. Bayes estimators for α (n=10, α = 0.5 and 1.5) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.40920(0.01879)	0.60398(−0.0401)	0.72079(−0.11182)
		PLF	0.43156(0.04470)	0.56981(−0.06833)	0.63853(−0.16452)
		LINEX (m = 0.6)	0.40338(0.00349)	0.61697(−0.00779)	0.75956(−0.02325)
		LINEX (m = 1.6)	0.39296(0.02599)	0.64414(−0.06425)	0.86409(−0.22926)
		LINEX (m = −0.6)	0.41464(0.00326)	0.59275(−0.00673)	0.69110(−0.01781)
		LINEX (m = −1.6)	0.42279(0.02173)	0.57697(−0.0432)	0.65397(−0.10691)
	1.2	SELF	0.43077(0.01103)	0.52703(−0.00250)	0.56593(−0.0020)
		PLF	0.44339(0.02524)	0.52274(−0.00857)	0.55158(−0.02870)
		LINEX (m = 0.6)	0.42740(0.00194)	0.52835(−0.00082)	0.57095(−0.00017)
		LINEX (m = 1.6)	0.42152(0.01479)	0.53040(−0.00539)	0.58041(−0.02317)
		LINEX (m = −0.6)	0.43401(0.00190)	0.52566(−0.00075)	0.56131(−0.00270)
		LINEX (m = −1.6)	0.43908(0.01329)	0.52330(−0.00595)	0.55436(−0.01850)
1.5	0.8	SELF	1.22762(−0.16114)	1.74298(−0.16283)	1.41716(0.07202)
		PLF	1.29468(0.06411)	1.66587(0.05037)	1.442349(0.06420)
		LINEX (m = 0.6)	1.17237(0.03314)	1.83872(0.04158)	1.38770(0.01767)
		LINEX (m = 1.6)	1.07148(0.24982)	2.18476(0.19804)	1.29963(0.18803)
		LINEX (m = −0.6)	1.27280(0.04712)	1.67533(0.04058)	1.43237(0.03131)
		LINEX (m = −1.6)	1.32299(0.15259)	1.59559(0.13581)	1.43751(0.12583)
	1.2	SELF	1.29259(0.09937)	1.54303(0.07560)	1.36701(0.07927)
		PLF	1.33047(0.07576)	1.53903(0.00799)	1.39570(0.05739)
		LINEX (m = 0.6)	1.26121(0.01882)	1.54462(0.01721)	1.34051(0.01589)
		LINEX (m = 1.6)	1.20553(0.13929)	1.53281(0.11634)	1.28623(0.12924)
		LINEX (m = −0.6)	1.32041(0.01869)	1.53785(0.01719)	1.38804(0.01620)
		LINEX (m = −1.6)	1.35680(0.10274)	1.52519(0.11214)	1.41174(0.09890)

Table 8. Bayes estimators for α (n=50, α = 0.5 and 1.5) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.4562(0.0060)	0.4425(0.0058)	0.4622(0.0062)
		PLF	0.4626(0.0131)	0.4494(0.0139)	0.4685(0.0125)
		LINEX (m = 0.6)	0.4542(0.0010)	0.4406(0.0011)	0.4604(0.0010)
		LINEX (m = 1.6)	0.4511(0.0078)	0.4375(0.0079)	0.4574(0.0077)
		LINEX (m = −0.6)	0.4578(0.0010)	0.4443(0.0011)	0.4640(0.0010)
		LINEX (m = −1.6)	0.4607(0.0075)	0.4474(0.0079)	0.4668(0.0072)
	1.2	SELF	0.4687(0.0029)	0.4620(0.0029)	0.4712(0.0028)
		PLF	0.4718(0.0062)	0.4652(0.0064)	0.4742(0.0061)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
		LINEX (m = 0.6)	0.4678(0.0005)	0.4611(0.0005)	0.4703(0.0005)
		LINEX (m = 1.6)	0.4663(0.0037)	0.4596(0.0038)	0.4688(0.0037)
		LINEX (m = −0.6)	0.4695(0.0005)	0.4629(0.0005)	0.4720(0.0005)
		LINEX (m = −1.6)	0.4710(0.0037)	0.4644(0.00381)	0.4734(0.0036)
1.5	0.8	SELF	1.3684(0.0544)	1.2017(0.0392)	1.3466(0.0556)
		PLF	1.3882(0.0395)	1.2179(0.0324)	1.3671(0.0410)
		LINEX (m = 0.6)	1.3517(0.0099)	1.1909(0.0064)	1.3298(0.0100)
		LINEX (m = 1.6)	1.3237(0.0071)	1.1774(0.0388)	1.3025(0.0705)
		LINEX (m = −0.6)	0.5074(0.0006)	0.4987(0.0006)	1.3630(0.0098)
		LINEX (m = −1.6)	1.4081(0.0635)	1.2401(0.0615)	1.3885(0.06714)
	1.2	SELF	1.4060(0.0264)	1.3253(0.0233)	1.3955(0.0267)
		PLF	1.4153(0.0187)	1.3341(0.0175)	1.4051(0.0191)
		LINEX (m = 0.6)	1.3979(0.0048)	1.318(0.0040)	1.3874(0.0048)
		LINEX (m = 1.6)	1.3844(0.0344)	1.3086(0.0267)	1.3741(0.0342)
		LINEX (m = −0.6)	1.4138(0.0046)	1.3326(0.0043)	1.4035(0.0047)
		LINEX (m = −1.6)	1.4261(0.0321)	1.3457(0.0325)	1.4162(0.0331)

Table 9. Bayes estimators for α (n=100, α = 0.5 and 1.5) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.50631(0.00287)	0.52968(0.00283)	0.53139(0.00272)
		PLF	0.50995(0.00620)	0.53235(0.00534)	0.53396(0.00512)
		LINEX (m = 0.6)	0.50519(0.00061)	0.52881(0.00052)	0.53056(0.00050)
		LINEX (m = 1.6)	0.50332(0.00404)	0.52729(0.00382)	0.52909(0.00036)
		LINEX (m = −0.6)	0.5074(0.00070)	0.4987(0.00060)	0.5095(0.00050)
		LINEX (m = −1.6)	0.50923(0.00467)	0.53182(0.00343)	0.53345(0.00329)
	1.2	SELF	0.50340(0.C	0.51379	0.5145
		PLF	0.50504(0.00327)	0.51522(0.00	0.5159
		LINEX (m = 0.6)	0.50291(0.00028)	0.51335(0.00025)	0.51411
		LINEX (m = 1.6)	0.50208(0.00212)	0.51259(0.00192)	0.51337(0.00189)
		LINEX (m = −0.6)	0.50390(0.00029)	0.51423(0.00	0.51498
		LINEX (m = −1.6)	0.50472	0.5149	0.5156
1.5	0.8	SELF	1.51	1.576	1.5285
		PLF	1.5294	1.55436	1.5853
		LINEX (m = 0.6)	1.50843	1.5451	1.56800
		LINEX (m = 1.6)	1.49159	1.52833	1.5283(0.04310)
		LINEX (m = −0.6)	1.52834	1.53821	1.58449(0.0058)
		LINEX (m = −1.6)	1.54375(0.04038)	1.56640(0.0390)	1.55306(0.03493)
	1.2	SELF	1.2	1.5	1.52
		PLF	1.5150	1.530	1.54
		LINEX (m = 0.6)	1.51455	1.52180(0.00260)	1.53180(0.00250)
		LINEX (m = 1.6)	1.49812	1.52450(0.01837)	1.51433(0.01889)
		LINEX (m = −0.6)	1.5	1.5172	1.5026
		LINEX (m = −1.6)	1.52170(0.01852)	1.52600(0.01721)	1.53696(0.01689)

Here, we compared the performance of the estimators based on their posterior risks. In comparison of Jeffrey's (non-informative prior) with Gamma and Gumbel (informative priors), informative priors give the better estimates of δ for all the loss functions. When comparing informative priors, we can see that the better estimates of δ and α are obtained when using the Gumbel type-II prior because the posterior risks are smaller for the loss functions considered here except SELF and LINEX with loss parameter 1.6. But for loss functions SELF and LINEX with loss parameter 1.6, the better estimates of δ and α are obtained through gamma prior. One can easily observe that the LINEX loss function with loss parameter 0.6 is a better choice of loss function for the estimation of α and δ. Since, corresponding posterior risks obtained are smaller for all the priors under consideration. Also, **Tables 4** to **9** depicts that the increase of sample size results in the decrease of posterior risks and the increase in parameter values result in the increase of the posterior risks.

4. Conclusion

In this paper, we studied the Bayesian analysis of Weibull distribution under different loss functions using priors both non-informative and informative priors. Here, we employed Lindley's approximation technique for the computation of Bayes estimators and the maximum likelihood estimators are obtained using MASS() package in RStudio. We conducted Monte Carlo simulation study to compare the different estimators based on posterior risks. We studied the effects of loss functions (both symmetric and asymmetric) by using Jeffrey's, Gamma and Gumbel type-II prior. From this study it is clear that for the estimation of δ and α Gumbel prior has smaller posterior risks when compared to the Jeffrey's and gamma priors. Also, based on posterior risks we observed that the LINEX loss function with $m = 0.6$ performs better than the other loss functions. So it is a better choice when considering loss functions. Hence, the combination of Gumbel type-II prior and LINEX loss function with $m = 0.6$ is the suggested combination for the parameter estimation in Weibull distribution.

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