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Research Article

Comparison of Parameter Estimation Methods in Weibull Distribution

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Abstract:

Introduction: The Weibull distribution is widely applied in reliability analysis, engineering, and life data modeling due to its flexibility. Accurate parameter estimation is essential for practical applications. This study aims to compare the effectiveness of classical and Bayesian approaches in estimating the scale and shape parameters of the Weibull distribution. **Methods**: Two parameter estimation methods are considered: Maximum Likelihood Estimation (MLE) and Bayesian estimation. Due to the non-existence of a conjugate prior for the Weibull parameters, Bayesian estimation is implemented using independent exponential priors for the scale and shape parameters. Approximate Bayesian estimators are computed using the Lindley approximation and the Markov Chain Monte Carlo (MCMC) method via the Metropolis-Hastings algorithm. A Monte Carlo simulation study is conducted using various sample sizes ($n = 20$, 50, 80, 110) to compare the performance of the methods based on the Mean Square Error (MSE) criterion. **Results**: Simulation results indicate that Bayesian methods generally outperform MLE in terms of lower MSE for small sample sizes. As sample size increases, all methods yield similar estimation performance. Both Lindley and MCMC methods provide reliable Bayesian estimates, with MCMC slightly outperforming Lindley in some cases. **Conclusions**: Bayesian estimation methods, particularly when applied with proper approximation techniques, can be considered effective alternatives to MLE for Weibull parameter estimation. These methods are especially advantageous in scenarios involving small sample sizes, where MLE may lack precision.

Keywords: Bayes estimator, Lindley Approximation, MCMC, Monte Carlo Simulation, Weibull Distribution.

1. Introduction

There are many applications for the Weibull distribution in statistics. It was first introduced by Waloddi Weibull in 1951 to predict the life span of machines. This distribution can be applied with two or three parameters depending on the field of use. This distribution is used in quality control, modeling of deterioration periods, analysis of life tables, availability of epidemic disease, determination of earthquake risk, definition of wind speed distribution and financial applications. Weibull distribution is commonly used in data sets related to failure rates [1]–[3]. It is a continuous and at the same time flexible distribution in this sense. Nowadays, this distribution is widely used in biology, engineering, quality control, seismic risk analysis, meteorological weather prediction models, radar systems modeling areas, wind speed distribution definition and many other field experiments.

Because of the wide applications area, it is very important to determine the best parameter estimation method for this distribution. Many authors have proposed various estimation methods for Weibull parameters. The least squares method, maximum likelihood method, moments method and Bayesian methods are used to estimate the parameters of the Weibull distribution. The maximum likelihood method is the most popular method. The efficiency of the maximum likelihood estimation method makes it popular. The least square method is computationally easier to handle and provides simple closed form solutions for the estimates. Hossain and Howlader (1996), made comparisons among several least squares and maximum likelihood estimator for complete samples and the shape parameter [4]. Ahmed et al. (2010) proposed Bayesian estimation with the Jeffrey's prior and extension of the Jeffrey's prior information for the Weibull parameters [5]. Hossain and Zimmer (2003) compared the maximum likelihood estimator to the least square estimator based on complete and censored samples [6]. Cox (1984) in [7] and Lawless (1982) in [8] made comparisons for censored data. Soland et al. (1969) introduce Bayesian analysis of the Weibull Process with unknown scale and shape parameters [9]. Guure and Ibrahim (2013) made comparisons for type-1 censored data [10].

While computing the Bayes estimates for the Weibull distribution [11]–[13], the continuous conjugate joint prior distribution of the shape and scale parameters does not exist and the closed form expressions of the Bayes estimators cannot be obtained. We must use approximation methods for this computation. In Bayesian approximation, the choice of prior distribution is very important. In this study, we assume that the scale parameter and the shape parameter both have the Exponential prior and they are independently distributed. We use the Lindley approximation and the Metropolis-Hasting algorithm, which is a method of Markov Chain Monte Carlo (MCMC), to obtain the approximate Bayes estimators. In simulation study we compare the effectiveness of the parameter estimation methods with Monte Carlo simulations.

In this study, we make comparison between the maximum likelihood and Bayes estimation of the standard parameterization form of Weibull distribution for the case of complete data [14]–[16]. The rest of the paper is organized as follows. In section 2, Weibull distribution is given. Section 3, maximum likelihood method is given to estimate the unknown parameters for Weibull Distribution. In section 4, Bayesian estimation method is investigated. Section 4.1, estimations of the unknown Weibull parameters are obtained by using Lindley approximation. In Section 4.2, the MCMC method is explained and in subsection 4.2.1, the Metropolis-Hasting algorithm is given. In Section 5, a simulation study is presented to evaluate the performances of the estimators. Section 6, we use real data set to illustrate the estimation procedure developed in section 3-4. The last section, we make some conclusion about parameter estimation methods for Weibull distribution.

2. Method:

Weibull Distribution

The Weibull distribution is a two-parameter (standard) distribution [17], generally α scale and β shape parameter. If a random variable X ~ Weibull (α, β) then its probability density function is defined as,

$$F(x;\alpha,\beta) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \quad 0 < x < \infty \quad \alpha > 0, \beta > 0 \tag{1}$$

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \quad 0 < x < \infty \quad \alpha > 0, \beta > 0 \tag{2}$$

where β is the shape and α is the scale parameter as the characteristic life parameter [18]. The expected value and the variance of the Weibull distribution are given,

$$E(X) = \beta\Gamma\left(1 + \frac{1}{\alpha}\right) \tag{3}$$

$$Var(X) = \beta^2\left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma^2\left(1 + \frac{1}{\alpha}\right)\right] \tag{4}$$

Maximum Likelihood Method

Maximum-likelihood estimation (MLE) is one of the most common parameter estimation methods for statistical models.

Suppose that $X1, X2, \ldots, Xn$ are independent and identically distributed $Weibull$ (α, β) random variables, where the parameters are assumed unknown. To estimate the parameters β and γ the maximum likelihood method is employed. The likelihood function of $X1, X2, \ldots, Xn$ can be constructed from Equation 1 as

$$f(x;\theta) = \prod_{i=1}^{n} f(x;\theta) \tag{5}$$

$$L(\theta;x) = f(x;\theta) \tag{6}$$

Logarithm of maximum likelihood equation like this

$$lnL\left(\hat{\theta};x\right) = \sum_{i=1}^{n} \ln f(x;\theta) \tag{7}$$

The log-likelihood function can be written as

$$L(\alpha,\beta) = \frac{\alpha^n}{\beta^{n\alpha}}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}} \tag{8}$$

It is found as:

$$lnL = nln\alpha - n\alpha ln\beta + (\alpha - 1)\sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} \tag{9}$$

Differentiating with respect to γ and β and equating to zero, the estimating equations are obtained

$$\frac{\partial lnL}{\alpha} = \frac{n}{\alpha} - nln\beta + \sum_{i=1}^{n} lnx_i - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} ln\frac{x_i}{\beta} = 0 \tag{10}$$

$$\frac{\partial lnL}{\partial\beta} = \frac{-n\alpha}{\beta} + \frac{\alpha}{\beta}\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha} = 0 \tag{11}$$

The MLE of parameters are obtained by solving the above nonlinear systems of equations. It is usually more convenient to use nonlinear optimization algorithms such as Newton Raphson to numerically maximize the log-likelihood function in Equation 9. In this study, we used multivariate Newton Raphson method to solve the Equations 10—11

Bayesian Estimation Method

Bayesian estimation method has received a lot of attention in recent times for analyzing failure time data, which has mostly been proposed as an alternative to that of the traditional methods. The Bayesian approach is based on Bayes' theorem, which was put forward by Thomas Bayes. In Bayesian method [19], [20], it is desirable to estimate the θ parameter using the $\boldsymbol{x} = x1, x2, \ldots, xn$ data for the statistical model defined by the probability (density) function $p(\boldsymbol{x}|\theta)$. In this method, the parameter is also considered as a random variable and therefore has its own distribution. If a prior knowledge about the parameter is not available, it is possible to make use of a non-informative prior distribution in Bayesian analysis

When both scale and shape parameters of the Weibull distribution are unknown and considered as random variables, Soland (1969) states that the Weibull distribution does not have a conjugate continuous joint prior distribution [9]. He suggests use of mixed prior distributions, discrete for the shape parameter, continuous for the scale parameter. Uniform prior for the shape parameter and Inverted Gamma prior for the scale parameter are proposed. Many different prior distributions are proposed for the shape and scale parameters such as Inverted Gamma- Compound Inverted Gamma, Discrete mass Function-Compound Inverted Gamma, Uniform Distribution-Compound Inverted Gamma, respectively. The Gamma prior on both the scale and shape parameters are considered. A simulation study is conducted for the both Gamma priors. A Gamma prior on the scale parameter and no specific prior on the shape parameter is assumed.

In this study we assume that, both the shape and scale parameters are unknown. Suppose the prior distribution β~$Exponential(1/a)$ for the shape parameter, the prior distribution α~$Exponential(1/b)$ for the scale parameter, and suppose that two parameters are independent of each other. Accordingly, the prior probability density functions for the parameters α and β are,

$$f(x|\alpha,\beta) = \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} exp\left\{-\left(\frac{x}{\beta}\right)^{\alpha}\right\},\ \ 0 < x < \infty \tag{12}$$

To be

$$L(x;\alpha,\beta) = \prod_{i=1}^{n} f(x_i|\alpha,\beta) = \frac{\alpha^n}{\beta^{n\alpha}} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}} \tag{13}$$

It is possible. On the other hand, the common prior probability density function of parameters α and β.

$$\pi(\alpha,\beta) = \pi(\alpha)\pi(\beta) \propto\ \alpha^{a-1}\beta^{c-1}e^{-d\beta-b\alpha},\quad \alpha,\beta,a,b,c,d \geq 0 \tag{14}$$

Since, X, X, ….. , X the joint probability density function of the parameters α and β is

$$f(x;\alpha,\beta) = L(x;\alpha,\beta)\,\pi(\alpha,\beta) =\ \alpha^{n+a-1}\beta^{c-1-na}e^{-(b\alpha+d\beta)}e^{-\beta^{-\alpha}\sum_{i=1}^{n}{x_i}^{\alpha}}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \tag{15}$$

It is given with, marginal probability density function of X,

$$f_x(x) = \int_{\alpha}\int_{\beta} f(x,\alpha,\beta)\,d\beta\, d\alpha \tag{16}$$

$$= \int_{\alpha}\int_{\beta} \alpha^{n+a-1}\beta^{c-1-na}e^{-(b\alpha+d\beta)}e^{-\beta^{-\alpha}\sum_{i=1}^{n}{x_i}^{\alpha}}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} d\beta\, d\alpha \tag{17}$$

then the likelihood function is proportioned to the marginal function and the joint posterior distribution of the two parameters is obtained as follows,

$$\pi(\alpha,\beta|x) = \frac{\alpha^{n+a-1}\beta^{c-1-na}e^{-(b\alpha+d\beta)}e^{-\beta^{-\alpha}\sum_{i=1}^{n}{x_i}^{\alpha}}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}}{\int_0^{\infty}\int_0^{\infty}\alpha^{n+a-1}\beta^{c-1-na}e^{-(b\alpha+d\beta)}e^{-\beta^{-\alpha}\sum_{i=1}^{n}{x_i}^{\alpha}}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} d\beta\, d\alpha} \tag{18}$$

As mentioned before, Bayesian estimators cannot be obtained analytically when the number of parameters is more than one. This situation can also be seen from Equation 18. Therefore, Bayesian estimation of the parameters will be obtained with lindley and MCMC method.

Lindley Approximation

Obtaining the Bayes estimator, expressed as the ratio of the two integrals, usually presents difficulties. Lindley (1980) developed the Lindley approximation method for the approximate solution of integrals forced in multi-parameter distributions when n is sufficiently large

$$\hat{u}(\theta) = E\big((u(\theta)|x)\big) \approx \Big[u(\theta) + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\big(u_{ij} + 2u_i\rho_j\big)\,\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{l=1}^{r} L_{ijk}\sigma_{ij}\sigma_{kl}u_l\Big]_{\hat{\theta}} \tag{19}$$

$$E(u(\alpha,\beta)|x) = u(\hat{\alpha},\hat{\beta}) + \hat{u}_{12}\hat{\sigma}_{12} + 0.5(\hat{u}_{11}\hat{\sigma}_{11} + \hat{u}_{22}\hat{\sigma}_{22}) + \hat{u}_1(\hat{\sigma}_{11}\hat{\rho}_1 + \hat{\sigma}_{21}\hat{\rho}_2) + \hat{u}_2(\hat{\sigma}_{12}\hat{\rho}_1 + \hat{\sigma}_{22}\hat{\rho}_2) + 0.5\left[\hat{L}_{111}(\hat{u}_1\hat{\sigma}_{11} + \hat{u}_2\hat{\sigma}_{11}\hat{\sigma}_{12}) + \hat{L}_{112}(3\hat{u}_1\hat{\sigma}_{11}\hat{\sigma}_{12} + \hat{u}_2(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2))\right] + 0.5\left[\hat{L}_{122}(\hat{u}_1(\hat{\sigma}_{11}\hat{\sigma}_{22} + 2\hat{\sigma}_{12}^2) + 3\hat{u}_2\hat{\sigma}_{12}\hat{\sigma}_{22}) + \hat{L}_{222}(\hat{u}_1\hat{\sigma}_{12}\hat{\sigma}_{22} + \hat{u}_2\hat{\sigma}_{22}^2)\right]$$

$$u_1 = \left.\frac{\partial u(\alpha,\beta)}{\partial \alpha}\right|_{\hat{\alpha},\hat{\beta}},\ u_2 = \left.\frac{\partial u(\alpha,\beta)}{\partial \beta}\right|_{\hat{\alpha},\hat{\beta}},\ u_{12} = \left.\frac{\partial^2 u(\alpha,\beta)}{\partial \alpha \partial \beta}\right|_{\hat{\alpha},\hat{\beta}},\ u_{21} = \left.\frac{\partial^2 u(\alpha,\beta)}{\partial \beta \partial \alpha}\right|_{\hat{\alpha},\hat{\beta}},\ u_{11} = \left.\frac{\partial^2 u(\alpha,\beta)}{\partial \alpha^2}\right|_{\hat{\alpha},\hat{\beta}},\ u_{22} = \left.\frac{\partial^2 u(\alpha,\beta)}{\partial \beta^2}\right|_{\hat{\alpha},\hat{\beta}}$$

$$\hat{L}_{111} = \left.\frac{\partial^3 L(\alpha,\beta)}{\partial \alpha^3}\right|_{\hat{\alpha},\hat{\beta}} = \frac{2n}{\hat{\alpha}^3} - \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right)^3 \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\alpha}}$$

$$\hat{L}_{112} = \left.\frac{\partial^3 L(\alpha,\beta)}{\partial \alpha^2 \partial \beta}\right|_{\hat{\alpha},\hat{\beta}} = \frac{2}{\beta} - \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right) \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\alpha}} + \frac{\alpha}{\beta} - \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right)^2 \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\alpha}}$$

$$\hat{L}_{122} = \left.\frac{\partial^3 L(\alpha,\beta)}{\partial \alpha \partial \beta^2}\right|_{\hat{\alpha},\hat{\beta}} = \frac{n}{\hat{\beta}^2} - \frac{(2\hat{\alpha}+1)}{\hat{\beta}^2} \sum_{i=1}^{n} \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\alpha}} - \frac{\hat{\alpha}(\hat{\alpha}+1)}{\hat{\beta}^2} \sum_{i=1}^{n} \ln\left(\frac{x_i}{\hat{\beta}}\right) \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\alpha}}$$

$$\hat{L}_{222} = \left.\frac{\partial^3 L(\alpha,\beta)}{\partial \beta^3}\right|_{\hat{\alpha},\hat{\beta}} = \frac{-2n\hat{\alpha}}{\hat{\beta}^3} + \frac{\hat{\alpha}(\hat{\alpha}+1)(\hat{\alpha}+2)}{\hat{\beta}^3} \sum_{i=1}^{n} \left(\frac{x_i}{\hat{\beta}}\right)^{\hat{\alpha}}$$

$$\hat{\rho}_1 = \left.\frac{\partial log\pi(\alpha,\beta)}{\partial \alpha}\right|_{\hat{\alpha},\hat{\beta}} = \frac{\alpha-1}{\hat{\alpha}} - b,\ \hat{\rho}_2 = \left.\frac{\partial log\pi(\alpha,\beta)}{\partial \beta}\right|_{\hat{\alpha},\hat{\beta}} = \frac{c-1}{\hat{\beta}} - d$$

MCMC Method

Following Bayes' rule

$$\boldsymbol{p(\theta|x) \propto L(\theta)\pi(\theta)}$$

For estimation of posterior distribution of Weibull distribution with standard parameterization we write the joint posterior distribution as

$$\boldsymbol{p(\alpha,\beta|x) \propto \frac{\alpha^n}{\beta^{n\alpha}} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\alpha}} \alpha^{a-1} \beta^{c-1} e^{-d\beta-b\alpha}}$$

we can use the MCMC to get the Bayesian estimates for Weibull parameters. MCMC is a general simulation method that replace analytic integration computations by summation over samples generated from iterative algorithms. The metropolis-Hastings algorithm is popular example of a MCMC method. In this study we used Metropolis-Hastings algorithm for multivariate distributions to obtain bayes estimates. We used component-wise updating approach.

Metropolis-Hastings Algorithm

The metropolis algorithm was first introduced in the field of statistical mechanics by metropolis et al., (1953) and later developed by hastings (1970). This algorithm is a very general MCMC method. Normalization constant it is used to obtain random samples from an arbitrarily complex proposed distribution of any dimension known as.

Generally, in cases where the number of parameters is large, the posterior distribution it is very difficult to create a sample. In this case, the metropolis- hasting method can be used to generate samples from distributions. With this method, samples can be easily generated from posterior distributions by using the steps given below.

Step1: $\theta^{(0)} = \left(\theta_r^{(0)}, \ldots, \theta_r^{(0)}\right)$ initial value is determined

Step2: j=1 is taken

Step3: $q\left(\theta_1^*\middle|, \theta_2^{(j-1)} \ldots, \theta_r^{(j-1)}, x\right)$ is generated from the proposed distribution

Step4: possibility of acceptance

$$p(\theta_1^{j-1},\theta^*) = min\left[1,\frac{\pi_1\left(\theta_1^*\middle|,\theta_2^{(j-1)}\dots,\theta_r^{(j-1)},x\right)q\left(\theta_1^{(j-1)}\middle|,\theta_2^{(j-1)}\dots,\theta_r^{(j-1)},x\right)}{\pi_1\left(\theta_1^{(j-1)}\middle|,\theta_2^{(j-1)}\dots,\theta_r^{(j-1)},x\right)q\left(\theta_1^*\middle|,\theta_2^{(j-1)}\dots,\theta_r^{(j-1)},x\right)}\right]$$

Is calculated

Step5: a number is generated from a uniform distribution in the range (0,1). Let it be this number

Step6: if u ≤ p $\left(\theta_1^{(j-1)},\theta^*\right)$ is $\theta_1^{(j-1)},\theta^*$, otherwise $\theta_1^{(j)},\theta_1^{(j-1)}$ is taken

Step7: take j=j+1 and repeat steps 1-5 until j=1, 2, …., N is done

Thus, a sample $\pi(\theta_1|\theta_2,\dots,\theta_r,x)$ is produced from the posterior distribution $\left(\theta_1^{(1)},\dots,\theta_1^{(N)}\right)$ by repeating similar steps, the posterior distribution of the parameters $\left(\theta_2,\theta_3,\dots,\theta_N\right)$ samples are produced.

3. Results and Discussion

Simulation Study

In this section, we conduct a Monte-Carlo simulation study to compare the performance of the classical and Bayesian estimation methods for Weibull distribution. For Bayesian estimation, we assume that both the shape and scale parameters have independent Exponential priors. We compute Bayesian estimates using Lindley's approximation and Metropolis-Hasting algorithm. We generated 1000 realizations of the Markov chains using Metropolis–Hastings algorithms. The convergence of the sequences of parameters for their stationary distributions is checked through different starting values. It is observed that after 100 burn-in periods, all the Markov chains reach their stationary condition. For all the numerical computations, we develop a program using Matlab 7 (R-2013). We also compute the maximum likelihood estimates and we compare these estimates results. In simulation study, we generate random data from Weibull distribution. The sample size is chosen as n = 20, 50, 80, 110. For each sample size, samples with β = 2 and γ = 2,3,4 values are generated by simulation. The mean square error for both parameters is chosen as the criterion to compare the performance of the maximum likelihood method and the estimation results E. Köksal Babacan, S. Kaya / Sigma J Eng & Nat Sci 38 (3), 1609-1621, 2020 1615 obtained with Bayesian methods. We use 1000 trial for simulation. Accordingly, the selected criterion, as an average measure of errors come from both parameters, is calculated as,

$$MSE = \frac{\sum_{i=1}^{1000}\left(\beta_i-\hat{\beta}_i\right)^2+(y_i-\hat{y}_i)^2}{1000}$$

Simulation results are given in Table 1. The Figures 1 to 4, represent the Weibull probability density function with the parameters of which we generate the data and with the parameters estimate via MLE, Lindley and MCMC procedures. Here we only report four group of density curve plots for $\boldsymbol{n} = \mathbf{20}, \mathbf{110}$ for parameter values $\boldsymbol{\beta} = \mathbf{2}, \boldsymbol{\gamma} = \mathbf{2}$ and $\boldsymbol{\beta} = \mathbf{2}, \boldsymbol{\gamma} = \mathbf{3}$. Table 1. Estimation results and MSE values.

Table 1. Estimation Results and MSE Values

Parameter Values	**Maximum Likelihood Method**				**Bayes Method: Lindley Approximation**			**Bayes Method: MCMC**		
	n	$\hat{\beta}$	$\hat{Y}$	**MSE**	$\hat{\beta}$	$\hat{Y}$	**MSE**	$\hat{\beta}$	Y	**MSE**
$\boldsymbol{\beta} = \mathbf{2}$ $\boldsymbol{Y} = \mathbf{2}$	**20**	1984113	2148155	0247663	2051442	2028711	0274917	2020395	2087577	0216158
	50	1995733	2057807	0079207	2016556	2022375	0074327	2008877	2035714	0075657
	80	2000999	2033285	0049009	2013351	2012337	0047336	2009365	2019934	0048222
	100	1994418	2025135	0031421	2003414	2009948	0030370	2000835	2015510	0030826
$\boldsymbol{\beta} = \mathbf{2}$ $\boldsymbol{Y} = \mathbf{3}$	**20**	1987253	3206099	0428115	2056101	3059351	0376660	2003225	3080299	0342021
	50	1998278	3084720	0139830	2023278	3032213	0131522	2004611	3040264	0128670
	80	1999780	3066825	0085441	2015161	3034573	0081446	2003028	3036147	0080368
	100	1996473	3025229	0062545	2007619	3002079	0061401	1998681	3002910	0060378
$\boldsymbol{\beta} = \mathbf{2}$	**20**	1992641	4342519	0844570	2061743	4157718	0723571	2000871	4125299	0616358

$Y = 4$	**50**	1997864	4124483	0234210	2024203	4055863	0217619	2000572	4042864	0205615
	80	1998724	4072200	0138386	2015006	4030102	0132553	2000463	4022842	0128528
	100	1997488	4049150	0107878	2009338	4018666	0104926	1998562	4013163	0102924

According to the simulation study results obtained in **Table 1**, it can be said that the results obtained with the maximum likelihood method and the Bayesian methods are similar. But for small sample size, the estimates using with Bayesian methods are better than the MLE. When sample size increases, the maximum likelihood method and the estimates obtained by Lindley approximation and Metropolis Hasting algorithm for the Bayesian methods are close to each other. When the number of sample size increases the Mean Square Error (MSE) decrease in all cases.

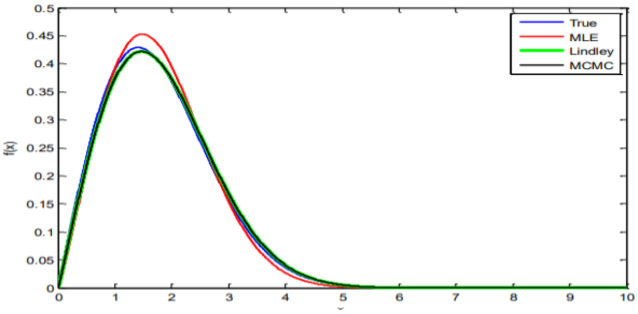

Figure 1. Weibull density curves (n = 20, β = 2, γ = 2)

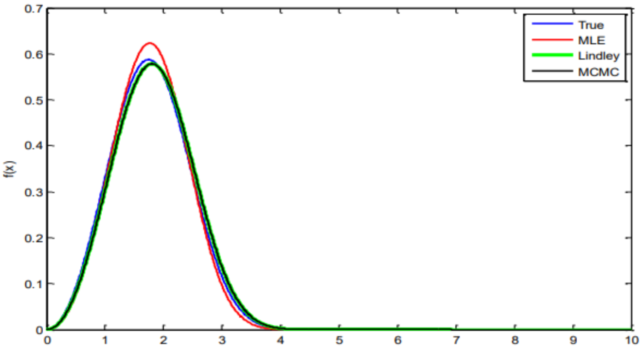

Figure 2. Weibull density curves (n = 20, β = 2, γ = 3)

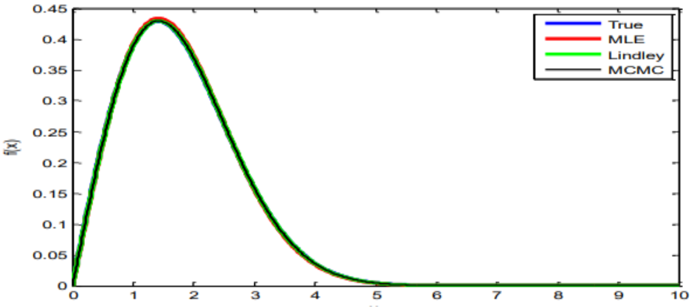

Figure 3. Weibull density curves (n = 110, β = 2, γ = 2)

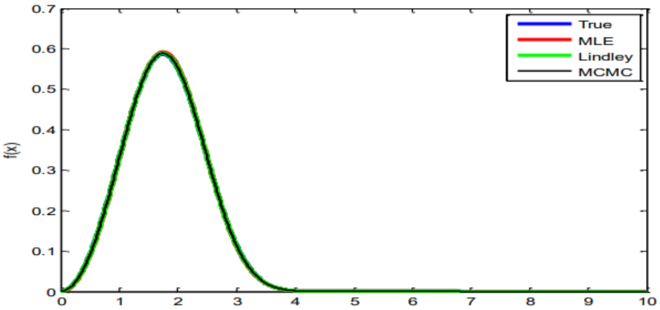

Figure 4. Weibull density curves ($n = 110$, $\beta = 2$, $\gamma = 3$)

From the density curves, for small sample size, the estimates with Bayesian methods are better fit than MLE. But, for big sample size all estimates are good fit to the real values.

4. Conclusion

In this paper, we use maximum likelihood estimation and Bayesian estimation for the two parameter Weibull distribution. MLE is one of the most frequently used parameter estimation methods. Newton-Raphson is one of the widely used methods for solving the system of equations especially in maximum likelihood estimation.

Bayesian estimation method receives a lot of attention in recent times. When we want to make conclusion via Bayesian method, if there isn't conjugate prior distribution, to get the posterior distribution has many difficulties. In this case, we need to use a numerical method or a MCMC method.

In this paper, Bayesian estimations are first obtained using Lindley approximation under the assumption of exponential priors while MLE are obtained using Newton-Raphson method. Second, Bayesian estimations are obtained using Metropolis-Hasting algorithm, which is a MCMC method.

A simulation study is conducted to examine and compare the performance of the estimates for different sample sizes with different values for parameters.

as a result of study, we can say that, Bayesian methods can be used as an alternative to the maximum likelihood method when the two parameters of the Weibull distribution are estimated. Especially, if sample size is small we can prefer to use the Bayesian estimation method.

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